B.sc. part-II paper IV

products of Three and Four vectors

(1) scalar Triple product: Def:- If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors then the product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called scalar triple product
$\vec{a}\cdot(\vec{b}\times\vec{c})$ is also represented by $[\vec{a}\,\vec{b}\,\vec{c}]$

(2) vector triple product of three vecters:- If $\vec{a}, \vec{b}, \vec{c}$ are three vectors then vector triple product of $\vec{a}, \vec{b}, \vec{c}$ be

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

(3) scalar product of Four vectors
Def:- If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four vecters then $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$ is called scalar product of four vectors and it represented as

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

(4) vecter product of Four vectors:- Def - If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four vectors then vector product of four vector defined as

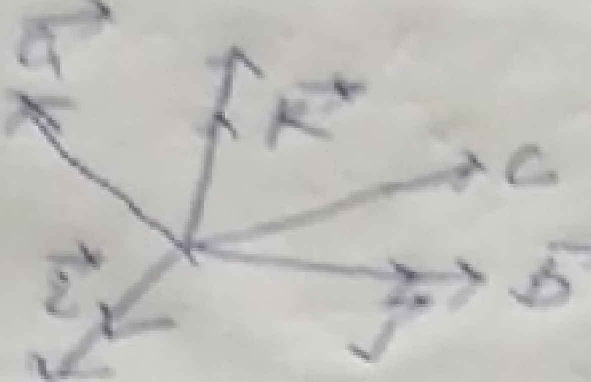
$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$$

$= [\vec{b}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$

Theorem :- prove that

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

proof :- let us take any arbitrary vector $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$



, vector $\vec{b}$ along $\vec{j}$ as $\vec{b} = b_2\vec{j}$ and $\vec{c}$ in the plane of $\vec{j}$ and $\vec{k}$ and $\vec{c} = c_2\vec{j} + c_3\vec{k}$

Then we have

$\vec{b}\times\vec{c} = b_2\vec{j}\times(c_2\vec{j} + c_3\vec{k})$

$\Rightarrow \vec{b}\times\vec{c} = b_2c_3\vec{j}\times k \quad (\because \vec{j}\times\vec{j} = 0)$

$= b_2c_3\vec{i} \quad [\because \vec{j}\times\vec{k} = \vec{i}]$

Thus $\vec{a}\times(\vec{b}\times\vec{c})$

$= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})\times b_2c_3\vec{i}$

$= 0 + a_2b_2c_3\vec{j}\times\vec{i} + a_3b_2c_3\vec{k}\times\vec{i} \quad (\vec{i}\times\vec{i} = 0)$

$\Rightarrow \vec{a}\times(\vec{b}\times\vec{c}) = -a_2b_2c_3\vec{k} + a_3b_2c_3\vec{j}$

$= a_2c_2b_2\vec{j} + a_3b_2c_3\vec{j} - a_2b_2c_3\vec{k} - a_2c_2b_2\vec{j}$

$= (a_2c_2 + a_3c_3)b_2\vec{j} - a_2b_2(c_2\vec{j} + c_3\vec{k})$

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

(11) Problem (1)

Show that $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}(\vec{a}\times\vec{b})=0$

Soln:- $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}(\vec{a}\times\vec{b})$

$=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}+(\vec{b}\cdot\vec{a})\vec{c}$
$-(\vec{b}\cdot\vec{c})\vec{a}+(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})b$

$=0$ [$\because \vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})c$ etc
[$\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ etc]

Problem (2) prove that $[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}]=[\vec{a}\,\vec{b}\,\vec{c}]$

Soln:- $[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}]$

$=(\vec{a}+\vec{b})\cdot[(\vec{b}+\vec{c})\times(\vec{c}\times\vec{a})]$

$=(\vec{a}+\vec{b})[\vec{b}\times\vec{c}+\vec{b}+\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$

$=\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})$
$+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{a})$

$=[\vec{a}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{b}\,\vec{a}]+[\vec{a}\,\vec{c}\,\vec{a}]$ [$\because \vec{c}\times\vec{c}=0$)
$+[\vec{b}\,\vec{b}\,\vec{c}]+[\vec{b}\,\vec{b}\,\vec{a}]+[\vec{b}\,\vec{c}\,\vec{a}]$

$=[\vec{a}\,\vec{b}\,\vec{c}]+[\vec{b}\,\vec{c}\,\vec{a}]$

($\because$ scalar triple product of three vector is zero if two of them equal)

$=[\vec{a}\,\vec{b}\,\vec{c}][\vec{a}\,\vec{b}\,\vec{c}]$

$=2[\vec{a}+\vec{b}+\vec{c}]$

problem (3) $[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}] = [\vec{a}\vec{b}\vec{c}]^2$

Soln:- $[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]$

$= (\vec{a}\times\vec{b})\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}$

$= (\vec{a}\times\vec{b})\{\vec{m}\times(\vec{c}\times\vec{a})\}$

[where $m = \vec{b}\times\vec{c}$]

$= (\vec{a}\times\vec{b})\{(\vec{m}\cdot\vec{a})\vec{c} - (\vec{m}\cdot\vec{c})\vec{a}\}$

$= \vec{a}\times\vec{b}\ \{(\vec{b}\times\vec{c}\cdot\vec{a})\vec{c} - (\vec{b}\times\vec{c}\cdot\vec{c})\vec{a}\}$

$= (\vec{a}\times\vec{b})\cdot[\vec{b}\ \vec{c}\ \vec{a}]\vec{c} - 0$

$= [\vec{a}\times\vec{b}\cdot\vec{c}]$ $[\because \vec{b}\times\vec{c}\cdot\vec{c} = 0]$ $[[\vec{b}\ \vec{c}\ \vec{a}]]$

$= [\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}]$

$= [\vec{a}\vec{b}\vec{c}]^2$

prob (4) Show that $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 0$

L.H.S. $= \{(\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{d}) - (\vec{b}\cdot\vec{d})(\vec{c}\cdot\vec{a})\}$
$+ (\vec{c}\cdot\vec{b})(\vec{a}\cdot\vec{d}) - (\vec{c}\cdot\vec{d})(\vec{a}\cdot\vec{b})\}$
$+ (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})\}$

$= 0$ [$\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ etc]

~~Problem ⑤ $\vec{i}\times(\vec{a}\times\vec{i})+(\vec{a}\times\vec{j})$~~

Problem ⑤ $\vec{i}\times(\vec{a}\times\vec{i})+\vec{j}\times(\vec{a}\times\vec{j})+\vec{k}(\vec{a}\times\vec{k})=2\vec{a}$

Soln:- L.H.S $=(\vec{i}\cdot\vec{i})\vec{a}-(\vec{i}\cdot\vec{a})\vec{i}+(\vec{j}\cdot\vec{j})\vec{a}-(\vec{j}\cdot\vec{a})\vec{j}+(\vec{k}\cdot\vec{k})\vec{a}-(\vec{k}\cdot\vec{a})\vec{k}$

[using $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$]

$=\vec{a}-(\vec{i}\cdot\vec{a})\vec{i}+\vec{a}-(\vec{j}\cdot\vec{a})\vec{j}+$ ~~$(\vec{k}\cdot\vec{k})\vec{a}$~~ $\vec{a}-(\vec{k}\cdot\vec{a})\vec{k}$

$[\because \vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1]$

$=3a-(\vec{i}\cdot\vec{a})\vec{i}-(\vec{j}\cdot\vec{a})\vec{j}-(\vec{k}\cdot\vec{a})\vec{k}$ — (1)

Let $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ — (2)

Then

$(\vec{i}\cdot\vec{a})\vec{i}=[i(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})]\vec{i}$

$=[a_1+0+0]\vec{i}$ $[\because \vec{i}\cdot\vec{j}=\vec{i}\cdot\vec{k}=0]$

$=a_1\vec{i}$

in the same way we have

$(\vec{j}\cdot\vec{a})\vec{j}=a_2\vec{j}$ and $(\vec{k}\cdot\vec{a})\vec{k}=a_3\vec{k}$

By

L.H.S. $=3\vec{a}-(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})$

$=3\vec{a}-\vec{a}$ (using 2)

$=2\vec{a}$